Variation of Entropy through a Shock Wave

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ALTHOUGH the velocity, pressure, temperature, and density vary monotonically through a shock wave, the entropy increases only at first, achieves a maximum within the shock structure, and then decreases to some final value. This nonmonotonic behavior of the entropy has been shown to be related to the heat conduction within the shock wave,[1] but this observation does not provide insight to its physical consequences, i.e., how does the heat conduction affect the shock structure such that the entropy decreases toward the rear of the wave. The purpose of this note is to present some results that connect the gasdynamics of the nonmonotonic entropy behavior to the heat conduction within the shock structure.

The entropy may be expressed as a function of the density and temperature as follows:

$$S - S_1 = C_v \ln \frac{T/\rho^{\gamma-1}}{T_1/\rho_1^{\gamma-1}} \tag{1}$$

where the subscripts refer to reference conditions. All symbols used in this analysis are standard notation. Using Eq. (1) and the continuity equation, $(d/dx)(\rho uA) = 0$, the density may be eliminated from Eq. (1) to give

$$\left\{\frac{dS/dx}{C_v} - (\gamma - 1)\frac{dA/dx}{A}\right\} = (\gamma - 1)\frac{du/dx}{u} + \frac{dT/dx}{T} \tag{2}$$

It is both advantageous and not unduly restrictive to assume that the total temperature of the flow is invariant. With this assumption, the energy equation may be written

$$C_p(dT/dx) + (udu/dx) = 0 \tag{3}$$

Equation (3) may be solved for dT/dx, and the result substituted into Eq. (2) to give

$$\left\{\frac{dS/dx}{C_v} - (\gamma - 1)\frac{dA/dx}{A}\right\} = (\gamma - 1)[1 - M^2]\frac{du/dx}{u} \tag{4}$$

where the relations $a^2 = \gamma RT$ and $R = C_p - C_v$ have been used.

This equation shows that during the course of any constant total temperature flow process in which the Mach number varies from some supersonic to some subsonic value (or the converse) the quantity within the braces must change sign. The equation therefore establishes the nonmonotonic entropy behavior where $(dA/dx) = 0$. It also shows that the magnitude of the entropy gradient at some point in any flow process depends upon the departure of the streamtube area variation from the isentropic streamtube area variation. This is a fundamental point with regard to the nonmonotonic entropy behavior within the shock wave (and to the entropy behavior in general).

Inspection of the subsonic region of the shock structure in light of the foregoing discussion shows why the entropy variation must necessarily be negative if nonzero. The streamtube area is invariant (for the case of a normal shock) while the streamtube area for constant entropy flow should increase. The constant area constraint causes a departure from isentropic conditions, and results in the entropy decrease toward the rear of the shock.

The roll of heat conduction in the nonmonotonic entropy behavior is determined as follows. The energy equation may be written

$$\rho u\frac{dE}{dx} + p\frac{du}{dx} - \frac{4}{3}\mu\left(\frac{du}{dx}\right)^2 - \frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \tag{5}$$

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This equation may be used in place of Eq. (3) to give the following expression:

$$\rho uT\left\{\frac{dS/dx}{C_v} - (\gamma - 1)\frac{dA/dx}{A}\right\} = \frac{1}{C_v}\left[\frac{4}{3}\mu\left(\frac{du}{dx}\right)^2 + \frac{d}{dx}\left(k\frac{dT}{dx}\right)\right] \tag{6}$$

It has been shown[1] that, at the front and rear points of the shock structure, the right-hand side of Eq. (6) will have the same sign as $(d/dx)[k(dT/dx)]$. Consequently, toward the rear of the shock structure, the quantity in the braces on the left will be negative for $k > 0$ and, hence, the entropy will be decreasing. In addition, Eq. (6) shows that for $k > 0$ and $dS/dx = 0$, $dA/dx > 0$. This means that the heat conduction causes a constant entropy streamtube to increase its area. It thus appears that the nonmonotonic entropy variation is due to a coupling of the heat conduction within the shock structure with the one-dimensional constraint on the compression process.

It should be noted that a rather complete treatment of the shock wave nonmonotonic entropy behavior exists in the literature.[2]

References

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Ionization of Trace Species in Slender Cone Boundary Layers

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Nomenclature

C^m = mass fraction of element m
C_m = mass fraction of species m
f = nondimensional stream function
F, G = Eq. (13)
I = Eq. (5)
k = forward ionization rate constant
K = Eq. (8)
l = $\rho\mu/\rho_w\mu_w$
M = molecular weight of mixture
r = radius of cone surface
Sc = Schmidt number
T = temperature
u, v = tangential, normal velocities in boundary layer
w = chemical mass generation rate
x, y = tangential, normal coordinate in boundary layer
Z = Eq. (13)
ϵ = Eq. (13)
η = $u_e r(2\xi)^{-1/2}\int_0^y \rho\, dy$
μ = viscosity
ρ = density
ξ = $\int_0^y \rho_w\mu_w u_e r^2 dx$

Subscripts

e = at edge of boundary layer
eq = equilibrium
max = maximum, or at maximum temperature
w = wall

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THE ionization process in the laminar hypersonic boundary layer of pointed slender cones is well understood.[1-3] For pure air, a complex series of chemical reactions takes place in a narrow high-temperature region of the boundary layer. Reaction products, which include ions and electrons, diffuse from this chemically active region toward the cold catalytic wall and the cold outer stream. At the tip of the cone, where the boundary layer is very thin, the flow is essentially frozen; further downstream, where the fluid particles have more time to spend in the high-temperature region of the flow, the peak concentrations of the reaction products increase gradually, approaching chemical equilibrium far downstream. In the early stages of this relaxation process (i.e., near the tip) the chemical processes are dominated by the "forward" reactions, the reverse reactions being insignificant due to the low concentrations of the reaction products. Levinsky and Fernandez[4] combined this fact with an ingenious approximation to account for the much higher speed of the so-called "exchange" reactions in reacting air and arrived at a "cubic" law for the variation of peak electron concentrations with distance from the cone tip. This cubic law agrees surprisingly well with exact numerical calculations.[1, 2] Discrepancies arise as equilibrium is approached and the effect of reverse reactions cannot any more be neglected.

The purpose of this note is to analyze ionization in the hypersonic boundary layer of slender cones for the case where a small amount of readily ionizable material is injected into the boundary layer. It will be assumed that this contaminant is much more readily ionizable than air, so that while contaminant ionization is in progress the air composition is unchanged from freestream. Ionization rates are assumed sufficiently small so that gross flow properties in the boundary layer are undisturbed. Both of these conditions are satisfied at high altitude re-entry of slender cones, where the ablation process is only in the initial stages and the ablating material contains trace amounts of alkali metals.

One may now write down the momentum and element conservation equations in transformed coordinates

$$f\frac{\partial}{\partial\eta}\left(\frac{u}{u_e}\right)+\frac{\partial}{\partial\eta}\,l\,\frac{\partial}{\partial\eta}\left(\frac{u}{u_e}\right)=0$$
$$f\frac{\partial C^m}{\partial\eta}+\frac{1}{Sc}\frac{\partial}{\partial\eta}\,l\,\frac{\partial C^m}{\partial\eta}=0 \tag{1}$$

Derivatives with respect to ξ (the so-called nonsimilar terms) have been neglected, and a constant Schmidt number has been assumed. Equation (1) and the boundary conditions imply an analogy between C^m and u/u_e:

$$\frac{C^m(\eta)-C^m(0)}{C_e{}^m-C^m(0)}=\frac{u}{u_e}\,(Sc\;\eta) \tag{2}$$

The boundary condition for element C^m at the wall depends on the injection rate and the mass fraction of material m, $C_w{}^m$, in the injected gas

$$\frac{(2\xi)^{1/2}v_w}{\mu_w r u_e}\,[C_w{}^m-C^m(0)]=-\frac{l}{Sc}\frac{\partial C^m}{\partial\eta}\bigg|_{\eta=0} \tag{3}$$

Equations (2) and (3) can be combined to solve for the concentration of element m at the wall

$$C^m(0)=\frac{x^{1/2}C_w{}^m+IC_e{}^m}{x^{1/2}+I} \tag{4}$$

where injection parameter I is defined by:

$$\frac{I}{x^{1/2}}\equiv\frac{\mu_w r}{(2\xi)^{1/2}v_w}\frac{\partial u}{\partial\eta}\bigg|_{\eta=0}=\left(\frac{3\mu_w}{2\rho_w u_e x}\right)^{1/2}\frac{1}{v_w}\frac{\partial u}{\partial\eta}\bigg|_{\eta=0} \tag{5}$$

and the second form of I in (5) is based on constant wall and freestream properties. Now the locally similar species conservation equation for the ions can be considered:

$$f\frac{\partial C_m{}^+}{\partial\eta}+\frac{2}{Sc}\frac{\partial}{\partial\eta}\,l\,\frac{\partial C_m{}^+}{\partial\eta}+\frac{2x}{3u_e}\frac{w_m{}^+}{\rho}=0 \tag{6}$$

where the factor of 2 accounts for ambipolar diffusion of ions and electrons. Since the ionization rate depends strongly on temperature and there are no other reaction products, the peak temperature and ion concentration will coincide at $\eta_{\max}$, at which point (6) simplifies to

$$\left\{\frac{2l}{Sc}\frac{\partial^2 C_m{}^+}{\partial\eta^2}+\frac{2x}{3u_e}\frac{w_m{}^+}{\rho}\right\}\bigg|_{\eta=\eta_{\max}}=0 \tag{7}$$

Examination of the numerical results of Refs. 1 and 3 reveals that, for a wide variety of conditions, the ratio

$$\left(-\frac{\partial^2 C_{\mathrm{NO}}{}^+}{\partial\eta^2}\bigg|_{\eta=\eta_{\max}}\right)\Big/C_{\mathrm{NO}}{}^+{}_{\max}=K \tag{8}$$

is a constant. Because of the similarity of the ionization processes in pure and contaminated air, observation (8) can be assumed to hold for $C_m{}^+$ as well.

The ionization process may be described by the following reaction

$$m+A\rightleftarrows m^+ + e^- + A \tag{9}$$

where A denotes air (i.e., N_2 or O_2) particles. The corresponding reaction rate is

$$\frac{w_m{}^+}{\rho}=\frac{k\rho}{M}\left[C_m-C_{m\,\mathrm{eq}}\left(\frac{C_m{}^+}{C_m{}^+{}_{\mathrm{eq}}}\right)^2\right] \tag{10}$$

Element conservation requires that

$$C_m=C^m-C_m{}^+ \tag{11}$$

Since $C_e{}^m$ is zero, Eqs. (2, 4, 7, 8, 10, and 11) may be combined at $\eta=\eta_{\max}$ to give the following quadratic equation for peak ion concentration in the boundary layer:

$$Z^2+F\,(1+1/\epsilon)\,Z-G=0 \tag{12}$$

where

$$\left.\begin{aligned}
&Z\equiv C_m{}^+{}_{\max}/C_m{}^+{}_{\mathrm{eq}}\\
&F\equiv C_m{}^+{}_{\mathrm{eq}}/C_{m\,\mathrm{eq}}\\
&G\equiv\frac{C^m(\eta_{\max})}{C_{m\,\mathrm{eq}}}=\frac{C_w{}^m}{C_{m\,\mathrm{eq}}}\left[1-\frac{u}{u_e}\,(Sc\;\eta_{\max})\right]\frac{x^{1/2}}{I+x^{1/2}}\\
&\epsilon\equiv\left[\frac{Sc}{3lK}\,\frac{x/u_e}{M/k\rho}\right]\bigg|_{\eta=\eta_{\max}}
\end{aligned}\right\} \tag{13}$$

The quantities in (13) which are functions of temperature must be evaluated at $T_{\max}$; the pressure is a constant, independent of both x and y in this problem. The four terms in (12) have the following physical significance: the quadratic term is due to the reverse reaction; the second term arises from depletion of the contaminant due to ionization; the third term is the diffusion term; and the last term is the forward reaction rate not considering effects of depletion. The solution of (12) is

$$Z=\frac{F}{2}\left(1+\frac{1}{\epsilon}\right)\left\{\left[1+\frac{4G}{F^2[1+(1/\epsilon)]^2}\right]^{1/2}-1\right\} \tag{14}$$

It may be observed from (13) that ϵ is the ratio of a typical flow time to a typical chemical time. In the near frozen regime, where approximation (8) and the assumption of frozen-air flow are most likely to hold, ϵ will be small and can be used as an expansion parameter. The result is

$$Z=\frac{G}{F}\,\epsilon\left\{1-\epsilon+\left[1-\frac{G}{F^2}\right]\epsilon^2+\cdots\right\}=\frac{G}{F}\left(\frac{\epsilon}{1+\epsilon}\right)\left\{1-\frac{G}{F^2}\left(\frac{\epsilon}{1+\epsilon}\right)^2+\cdots\right\} \tag{15}$$

In the second form of (15), the effect of depletion is accounted for explicitly in the leading term by an appropriate change of the expansion parameter. Numerical solutions of (7) for cesium and sodium ionization,[5] which were based on five ionized trace species (NO^+, Cs^+, or Na^+, O_2^-, O^-, e^-), five neutral air species (O_2, N_2, O, N, NO), and a ten-reaction system for the ionization process, were compared with (15). The comparison showed that the leading term of the second form of (15) agreed within 5% with all the results of the more elaborate calculations over a range of 0 to 99.5% ionization. Explicitly

$$\frac{C_{m^+\,\max}}{C^m(\eta_{\max})} = \left\{1.0 + \frac{3lKu_e}{Sc\rho x(k/M)_{\text{eff}}}\right\}^{-1.0} \qquad (16)$$

for both cesium and sodium. The effective forward ionization rate in (16) combines three processes: a general neutral particle, O_2, and O.

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An Extension of Hetényi's Method through the use of Macaulay Brackets

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THERE seems to be a continuing interest in singularity functions as applied to beam and plate bending. In this journal, a recent article by Urry[1] extends the theory to beam columns, and a more comprehensive survey of the field by Weissenburger[2] traces the historical development of the subject. It should be noted in passing, that the work of Csonka[3, 4] treats the very general problem of an ordinary differential equation with constant coefficients whose right-hand side is composed of linear combinations of singularity functions of the Macaulay type. The example given by Urry can be handled as a special case by the method developed by Csonka, who solves this particular problem as an example. More recently Pilkey[5] has given an excellent account of the engineering significance of the method in an article that contains an exhaustive bibliography on the subject. Pilkey[6] has also introduced the theory of distribution of Laurent Schwartz to set the concept of singularity functions as used in bending theory on a sound mathematical basis. The purpose of this note is to extend the work of this writer[7–9] to the method of Hetényi.[10]

An example will serve to illustrate the application of the method. Consider a beam cantilevered at A, of total length $2a$, such that $\overline{AB} = \overline{BC} = a$. Let the loading on the beam be given as

$$0 < x < a \qquad w(x) = 0$$

$$a < x < 2a \qquad w(x) = (wo/a)(x - a)$$

where the origin is at A, and the end C is free.

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Following Hetényi, the deflection curve of part AB of the beam can be expanded into the following Maclaurin series:

$$y_{AB}(x) = y(0) + y'(0)x + y''(0)x^2/2! + \ldots + y^{(n)}(0)x^n/n!$$

The expansion being valid in the range $0 < x < a$.

Similarly, a Taylor expansion gives the deflection curve of part BC as

$$y_{BC}(x) = y(a) + y'(a)(x - a) + \ldots + y^{(n)}(a)(x - a)^n/n!$$

which is valid in $a < x < 2a$.

The Bernoulli-Euler equation gives

$$y''(x) = [M(x)/EI]$$

from which it follows that

$$y'''(x) = [V(x)/EI] \qquad y^{\text{IV}}(x) = [w(x)/EI] \qquad \text{etc.}$$

recalling the relations

$$dM/dx = V \qquad dV/dx = w$$

The Maclaurin expansion now reduces to

$$y_{AB}(x) = (M_A x^2/2EI) + (V_A x^3/6EI)$$

after the initial conditions $y(0) = y'(0) = 0$ have been inserted.

The Taylor expansion now becomes

$$y_{BC}(x) = y(a) + y'(a)(x - a) + \frac{M_B}{2EI}(x - a)^2 + \frac{V_B}{6EI}(x - a)^3 + \frac{w_0}{120\,EIa}(x - a)^5$$

where V_A, M_A, V_B, M_B, and w are assumed to act in the positive direction.

Equilibrium of the section AB of the beam requires that

$$M_A + V_A a = M_B \text{ and } V_A = V_B$$

The continuity of the slopes and deflections at B requires that

$$y'_{AB}(a) = y'_{BC}(a) \text{ and } y_{AB}(a) = y_{BC}(a)$$

This yields

$$y_{AB}(a) = \frac{M_A a^2}{2EI} + \frac{V_A a^3}{6EI} = y_{BC}(a)$$

$$y'_{AB}(a) = \frac{M_A a}{EI} + \frac{V_A a^2}{2EI} = y'_{BC}(a)$$

Rewriting the Taylor series after substituting the equilibrium and compatibility conditions leads to

$$y_{BC}(x) = \frac{M_A a^2}{2EI} + \frac{V_A a^3}{6EI} + \left(\frac{M_A a}{EI} + \frac{V_A a^2}{2EI}\right)(x - a) + \frac{(M_A + V_A a)}{2EI}(x - a)^2 + \frac{V_A}{6EI}(x - a)^3 + \frac{w_0}{120EIa}(x - a)^5$$

which, after some algebra, reduces to

$$y_{BC}(x) = \frac{M_A x^2}{2EI} + \frac{V_A x^3}{6EI} + \frac{w_0}{120\,EIa}(x - a)^5$$

This in turn can be written as

$$y_{BC}(x) = y_{AB}(x) + (w_0/120\,EIa)(x - a)^5$$

It is now clear that the use of Macaulay brackets can now be introduced to write

$$y(x) = \frac{M_A x^2}{2EI} + \frac{V_A x^3}{6EI} + \frac{w_0}{120\,EIa}\langle x - a\rangle^5$$

which is an expression now valid for the entire beam. This expression could have been written by inspection, starting